Homework 1, part 2/2

Version as of Oct 2, 2024, a few typos and errors fixed

Due date for entire HW1 is Oct 20. Either paper or electronic submission is fine.

Each exercise is worth 1 point. Extra point for each reported error! – please use forum.

Total score for this part is #points/#total, so the maximum is 1 (more with extra credit)

We encourage you to use QuTiP in Python to do some of these exercises, at least to verify your answers or just to explore the problem without having to do the math by hands.

Suggested study material:
- L. Susskind Theoretical Minimum book, Ch. 4
- Kay, Laflamme, Mosca Quantum Computing book, Ch 3.1-3.2
- Wiki is always there for you:

`https://en.wikipedia.org/wiki/Two-state_quantum_system`

- We encourage the use of QuTiP to do some of these exercises and gain intuitition without excessive math workout.

`https://qutip.org/qutip-tutorials/`

A. Time-evolution of qubit state

Time-dependent Schrodinger's equation. A quantum state $|\Psi\rangle$ evolves in time according to the Schrodinger's equation:

$$\frac{\partial|\Psi\rangle}{\partial t} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle \tag{1}$$

where we are introducing the Planck constant(s) $h = 2\pi\hbar$ and the Hamiltonian operator $\hat{H}$, the quantum version of the Hamiltonian function in classical mechanics. The meaning of this equation can be understood by considering the state vectors of the system at time t and at a very close time $t + \Delta t$. That is,

$$|\Psi(t+\Delta t)\rangle = |\Psi(t)\rangle - \frac{i\Delta t}{\hbar}\hat{H}|\Psi(t)\rangle = \Big(\hat{I} - \frac{i\Delta t}{\hbar}\hat{H}\Big)|\Psi(t)\rangle. \tag{2}$$

Since $\langle\Psi(t+\Delta t)|\Psi(t+\Delta t)\rangle = \langle\Psi(t)|\Psi(t)\rangle = 1$, we must insist that $\hat{I} - \frac{i\Delta t}{\hbar}\hat{H}$ is unitary, which in turn means $\hat{H}^\dagger = \hat{H}$, that is Hamiltonian is a hermitian operator (verify both conclusions). This conclusion agrees with the fact that $\hat{H}$ in the Schrodinger's equation represents the system's total energy and hence it should be an observable, and observables must indeed be described by hermitian operators (otherwise they can't be measured). To solve the Schrodinger's equation we can split the evolution time interval into many small intervals Δt, and recover $|\Psi(t)\rangle$ from $|\Psi(t=0)\rangle$ by repeatedly applying relation (2). We can also run the evolution backwards in time, and find out what $|\Psi(t\rangle)$ was before $t = 0$. We say the evolution is reversible.

Exercise 1: Verify the following equation for the time evolution of the *mean value* of a measurement outcome of some hermitian operator $\hat{L}$:

$$\frac{\partial}{\partial t}\langle\Psi(t)|\hat{L}|\Psi(t)\rangle = \langle\Psi(t)|\Big(-\frac{i}{\hbar}[\hat{L},\hat{H}]\Big)|\Psi(t)\rangle, \tag{3}$$

where $[\hat{L},\hat{H}] = \hat{L}\hat{H} - \hat{H}\hat{L}$ is called the commutator of operators $\hat{L}$ and $\hat{H}$. For a mean value of a quantum observable to change, its operator must not commute with the Hamiltonian operator.

Time-dependence of $|\Psi\rangle$ becomes especially simple in the basis of the eigenstates of $\hat{H}$. Let's take a simple qubit example with the Hamiltonian operator written as $\hat{H} = E_0|E_0\rangle\langle E_0| + E_1|E_1\rangle\langle E_1|$, where E_0 and E_1 are eigenvalues and $|E_0\rangle$ and $|E_1\rangle$ are the corresponding eigenvectors of $\hat{H}$. Then we can write an arbitrary qubit state as

$|\Psi(t)\rangle = \alpha_0(t)|E_0\rangle + \alpha_1(t)|E_1\rangle.$

Exercise 2: Plug the above wave function into Eq. 1 and obtain the following solution of the Schrodinger's equation:

$$|\Psi(t)\rangle = \exp(-iE_0t/\hbar)\times\alpha_0(t=0)|E_0\rangle + \exp(-iE_1t/\hbar)\times\alpha_1(t=0)|E_1\rangle. \tag{4}$$

Hint: you can simply plug the above solution into Eq. 1

Remember, the global phase is not important, so we can move the time-dependence entirely to the α_1-amplitude. The time dependence is given entirely by the energy difference of the two qubit states:

$$|\Psi(t)\rangle = \alpha_0(t=0)|E_0\rangle + \alpha_1(t=0)\exp(-i(E_1-E_0)t/\hbar)|E_1\rangle \tag{5}$$

Exercise 3: Show that if a qubit starts in an energy eigenstate $|E_0\rangle$ or $|E_1\rangle$, it stays in that state, no time-evolution takes place.

Exercise 4: Set up a Bloch sphere in the basis $|E_0\rangle$, $|E_1\rangle$ and consider $\alpha_0(t=0) = \alpha_1(t=0) = 1/\sqrt{2}$. Show the qubit state at $t=0$ and mark its time-evolution.

We finally have a quantum expression that has some physical meaning. Namely, the phase in Eq. (5) oscillates at the angular frequency $(E_1-E_0)/\hbar$, and the period of oscillations is $2\pi\hbar/(E_1-E_0) = h/(E_1-E_0)$.

Exercise 5: Let's assume we have such a macroscopic qubit that the energy difference $E_1 - E_0$ is given by the nutritional value of Big Mac (590 food calories). Each food cal. is 4200 Joules, which is the SI unit measure of energy. In the same units, $h \approx 6 \times 10^{-34}$ [Joule second]. Question: what would be the period of oscillations of the qubit phase in this case? [just divide the two numbers] Does it make sense why Big Mac is classical now? How small should the qubit energy difference be for the phase to oscillated in time at a more experimentally accessible frequency?

Exercise 6: Show that a modified Hamiltonian $\hat{H} - \hat{I}E_0$ (where E_0 is the lowest eigenvalue of $\hat{H}$) has the same eigenvectors as $\hat{H}$ and the same time-evolution of the qubit state.

Another way to solve the Schrodinger's equation (for a time-independent $\hat{H}$) is to simply guess that:

$$|\Psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\Psi(t=0)\rangle \tag{6}$$

Exercise 7: Check that the proposed above evolution operator indeed solves the Schrodinger's equation (1). Then use the matrix exponentiation trick for $\hat{H} = E_0|E_0\rangle\langle E_0| + E_1|E_1\rangle\langle E_1|$ to verify the solution in Eq. 5

Time-independent Schrodinger's equation. All that is left to do for solving the regular Schrodinger's equation is to find the eigenstates and eigenvalues of the $\hat{H}$ operator. This leads to the so-called time-independent Schrodinger equation:

$\hat{H}|E\rangle = E|E\rangle$

B. Hamiltonian operator for qubits

Just like in classical mechanics, we can think of a quantum system's Hamiltonian as what rigorously defines this particular system. Every instrument comes with a manual, every physical quantum system comes with a Hamiltonian. Curiously, the options for a legal Hamitonian operator for a two-level system are so limited that we can just guess it. Indeed, we know that the Hamiltonian matrix (say in the computational basis) must be hermitian, which limits its entries to 4 independent numbers: $\hat{H}/\hbar = \begin{pmatrix} a & x-iy \\ x+iy & b \end{pmatrix}$ where a, b, x, y are all real numbers.

Exercise 8: Show by an explicit demonstration that any Hermitian 2×2 matrix can be written as a superposition of

$$\hat{H}/\hbar = -\omega_I \hat{I} - \omega_X \hat{X}/2 - \omega_Y \hat{Y}/2 - \omega_Z \hat{Z}/2, \tag{7}$$

where $\omega_0, \omega_X, \omega_Y, \omega_Z$ are all real numbers and $\hat{X}, \hat{Y}, \hat{Z}$ are the Pauli matrices. The minus signs is just a convention, a part of the definition of ω's The term $\hat{I}$ has no significance as it simply shifts all the eigenvalues by ω_I, so we don't loose any information by setting $\omega_I = 0$.

Note, Eq. 7 is the most general Hamiltonian for a stationary qubit (or spin). In case of the real spin, say of an electron in a quantum dot, the frequencies $\omega_X, \omega_Y, \omega_Z$ are proportional to magnetic fields along the three axis. In artificial systems, like superconducting qubits, they have nothing to do with magnetic fields. But because of the electron spin analogy, the three coefficients are often called "magnetic fields" or just "fields".

It is quite convenient that $H/\hbar$ has units of frequency (angular frequency), which is a relatively intuitive physical quantity. So in what follows we would always quote $\hat{H}/\hbar$ instead of $\hat{H}$, which would effectively eliminate $\hbar$ from our equations, unless we go for a deeper journey into the microscopic origins of the Hamiltonian parameters, for which we are not ready yet.

Exercise 9: Show that eigenvalues of $\hat{H}$ in Eq. 7 are given by $\pm\sqrt{\omega_X^2 + \omega_Y^2 + \omega_Z^2}/2$

Exercise 10: Show that eigenstates of $\hat{H}$ in Eq. 7 in the Bloch sphere are aligned along (or against) the axis defined by $(\omega_X, \omega_Y, \omega_Z)$.

Hint: $\omega_Z = \sqrt{\omega_Z^2 + \omega_X^2 + \omega_Y^2} \times \cos\theta$, $\omega_X = \sqrt{\omega_Z^2 + \omega_X^2 + \omega_Y^2} \times \sin\theta\cos\phi$, $\omega_Y = ...$, apply $\hat{H}$ to the suspect eigenvector $|\Psi\rangle = \cos\theta/2|0\rangle + ...$ and use some trigonometry.

Exercise 11: Another way to guess the eigenstates of $\hat{H}$ is by applying a rotation operator to states $|0\rangle$ and $|1\rangle$ to align them along the direction $(\omega_X, \omega_Y, \omega_Z)$. This can be done in sequence, first rotate by θ in the Z-X plane and then add the ϕ-rotation in the X-Y plane. Create the rotation operators in terms of $\omega_X, \omega_Y, \omega_Z$, create the suspect eigenstates by applying the rotation operators to $|0\rangle$ and $|1\rangle$ and check if the rotated basis states are indeed the eigenstates.

The outcome of the last two exercises is important: they show that the dynamics of a qubit with the most general possible Hamiltonian is equivalent to a much simpler Hamiltonian of $\hat{H} = \frac{1}{2}\omega\hat{Z}$, if we change the computational basis to states along and against the $(\omega_X, \omega_Y, \omega_Z)$ vector in the Bloch sphere.

C. Commutation relations of $\hat{X}$, $\hat{Y}$, $\hat{Z}$

Let us now play a bit with the Eq. 3 which describes the evolution of the mean values of the observables of our qubit. Using this equation requires us finding the commutation relation of $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ operators. Remember, these are just Pauli matrices. So:
$[\hat{X}, \hat{Z}] = \hat{X}\hat{Z} - \hat{Z}\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = -2i\hat{Y}$
Likewise
$[\hat{Y}, \hat{Z}] = 2i\hat{X}$
$[\hat{Z}, \hat{Z}] = 0$
...

Exercise 12: Verify all the other commutation relations involving Pauli matrices by directly multiplying the corresponding matrices. Feel free to do it on a computer.

Everything you possibly wanted to (or did not) know about Pauli matrices is here:

`https://en.wikipedia.org/wiki/Pauli_matrices`

Let us take the most general possible qubit Hamiltonian given by Eq. 7 and write down the time evolution equation for the expectation values:

$\langle\dot{X}\rangle = \langle(-i/\hbar)[\hat{X}, \hat{H}]\rangle = -\omega_Z\langle\hat{Y}\rangle + \omega_Y\langle\hat{Z}\rangle$
$\langle\dot{Y}\rangle = \langle(-i/\hbar)[\hat{Y}, \hat{H}]\rangle = -\omega_X\langle\hat{Z}\rangle + \omega_Z\langle\hat{X}\rangle$
$\langle\dot{Z}\rangle = \langle(-i/\hbar)[\hat{Z}, \hat{H}]\rangle = -\omega_Y\langle\hat{X}\rangle + \omega_X\langle\hat{Y}\rangle$

These three equations highlight that the reason the mean value of an observable changes in time is that its underlying operator does not commute with the Hamiltonian operator. Denoting the mean values of the spin operators as $\langle\hat{X}\rangle = x$, $\langle Y\rangle = y$, $\langle Z\rangle = z$, we get an equation:

$$\dot{\vec{r}} = \hat{M}\vec{r}, \quad \hat{M} = \begin{pmatrix} 0 & -\omega_Z & \omega_Y \\ \omega_Z & 0 & -\omega_X \\ -\omega_Y & \omega_X & 0 \end{pmatrix} \tag{8}$$

The solution is quite familiar to us by now: $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \exp(\hat{M}t) \begin{pmatrix} x(t=0) \\ y(t=0) \\ z(t=0) \end{pmatrix}$

Exercise 13: Let's consider a simplified case of $\hat{H}/\hbar = -\omega\hat{Z}/2$. Describe the evolution of $z(t)$ which follows from Eq. 8.

Exercise 14: Continuing from the previous exercise, note that the Eq. 8 for the evolution of $x(t)$ and $y(t)$ is exactly the same as the equation of motion for the harmonic oscillator. Show the time-evolution in the x-y plane (equatorial plane of the Bloch sphere).

Exercise 15: Set $\omega_X = \omega_Y = \omega_Z = 1$ and solve the equation of motion for x, y, z numerically using any solver you have available. You basically need to exponentiate the matrix $\hat{M}$. Demonstrate that the solution is a rotation of the initial (x, y, z)-vector around the vector oriented along $(1, 1, 1)$.

Note, the evolution equation for the mean values x, y, z of the spin projection operators in quantum mechanics are reminiscent of the motion of a classical gyroscope. Check out this video and try to identify the analogy

`https://www.youtube.com/watch?v=8H98BgRzpOM`

This spin-gyroscope analogy is very helpful in guessing the solutions of the Schrodinger's equation for the evolution of state vector in the Bloch sphere. And it helps to understand why we call this thing "spin"!

D. Changing reference frames: "rotating" frame vs. "lab" frame

Check out this brief note about the frame change in quantum mechanics:

`https://en.wikipedia.org/wiki/Unitary_transformation_(quantum_mechanics)`

Exercise 16: Consider a qubit with a Hamiltonian $\hat{H}/\hbar = -\omega\hat{Z}/2$. Consider that at time $t = 0$ the qubit was initialized in state $|\Psi_0\rangle = |0\rangle/\sqrt{2} + |1\rangle/\sqrt{2}$. Show that the time evolution of the qubit state would be

$$|\Psi(t)\rangle = |0\rangle/\sqrt{2} + \exp(-i\omega t)|1\rangle/\sqrt{2}.$$

This is initially $|+\rangle$ state vector rotating in the equatorial plane of the Bloch sphere at an angular rate ω.

Exercise 17: Apply a unitary transformation $\hat{U} = \exp(-i\omega\hat{Z}/2)$ to the time-dependent state of the qubit in the previous exercise. What new Hamiltonian describes the time-evolution of this new state?

Here is the general idea about changing frames. Suppose that in the "lab frame" (no frame changes yet), we have a system (think 1 qubit) which is described by a Schrodinger's equation $|\dot{\Psi}\rangle = (-i/\hbar)\hat{H}|\Psi(t)\rangle$. Let us take any unitary transformation $\hat{U}$ and define a new state $|\Psi'\rangle = \hat{U}|\Psi\rangle$. What is the Schrodinger equation for $|\Psi'\rangle$? We use relations $U^\dagger U = I$ and $U^{-1} = U^\dagger$ and with a bit of matrix multiplication, we get the following equation: $|\dot{\Psi}'\rangle = (-i/\hbar)H'|\Psi'\rangle$, where the new Hamiltonian is $H' = UHU^\dagger + i\hbar\dot{U}U^\dagger$. It's ok to use time-dependent U as well, otherwise $\dot{U} = 0$. With a clever choice of U, the matrix H' can be much simpler than H, which makes the qubit dynamics more transparent. Let's keep this trick in mind.

Exercise 18: Consider a qubit with a Hamiltonian $H/\hbar = -\omega_0\hat{Z}/2$. Consider a frame change defined by the unitary $\exp(-i\omega t\hat{Z}/2)$. What is the Hamiltonian in this new frame? How would you choose ω to make the evolution as simple as possible?

E. Implementing quantum gates: evolution after a fast switch

Time to discuss how one might implement the rotations of the qubit state in the Bloch sphere. Such rotations are often called gates, as they change a given qubit state into a new one according to some fixed rule (the rotation). The first common approach is to utilize abrupt switching of the fields $\hat{X}$, $\hat{Y}$, $\hat{Z}$. For any time-independent Hamiltonian $\hat{H}$, the state evolution from times t_1 to time t_2 is given by $|\Psi(t_2)\rangle = \hat{U}(t_1, t_2)|\Psi(t_1)\rangle$, where $\hat{U} = \exp(-i\hat{H}(t_2 - t_1)/\hbar)$. So, for the most general qubit Hamiltonian, we get $\hat{U}(t_1, t_2) = \exp(-i(t_2 - t_1)(\omega_X\hat{X} + \omega_Y\hat{Y} + \omega_Z\hat{Z}))$. Switching on or off one of the values ω_X, ω_Y, ω_Z modifies the evolution operator. So to find the quantum state after a number of such "switches" we just apply the corresponding evolution operators operators to account for the evolution at every interval of time in between the switches.

Exercise 19: Consider a qubit with a Hamiltonian $H = 0$ prepared in state $|0\rangle$. Let's say an experimentalist can turn on and off the fields $\omega_X, \omega_Y, \omega_Z$ in the Hamiltonian (7) abruptly at any time. So, say at time $t = 0$ we abruptly turn on the Y-field, which means the Hamiltonian becomes $H/\hbar = -\omega\hat{Y}/2$. The qubit at this moment is still in state $|0\rangle$, which is no longer an eigenstate of the new Hamiltonian. So some time-evolution will take place. Show this evolution on the Bloch sphere. At what time should the experimentalist set $\omega = 0$ to arrive at states $|1\rangle$, $|+\rangle$, $|-\rangle$?

Exercise 20: Let's do the same exercise as before, but with the following change: at time $t = 0$ we abruptly turn on the X-field, which means the Hamiltonian becomes $H/\hbar = -\omega\hat{X}/2$. Show this evolution on the Bloch sphere. At what time should the experimentalist set $= 0$ to obtain states $|1\rangle$, $|+i\rangle$, $|-i\rangle$?

Exercise 21: Let's do a slightly fancier manipulation. We take a hamiltonian $\hat{H}/\hbar = -\omega\hat{Z}/2$ and start again in state $|0\rangle$. At time $t = 0$ we turn on an X-field, which makes the new Hamiltonian $\hat{H}/\hbar = -\omega\hat{Z}/2 - \omega\hat{X}/2$ (same strength field along both axis). Solve for the evolution of the state for $t > 0$ and show it on the Bloch sphere. At what time $|0\rangle$ turns into $|+\rangle$?

Exercise 22: In the previous exercise, repeat the calculation but start from state $|+\rangle$ at $t = 0$. At what time $|+\rangle$ turns into $|0\rangle$?

Note, the result of the previous two exercises can be used to create the Hadamard operation (gate), which turns $|0\rangle \to |+\rangle$ and $|1\rangle \to |-\rangle$ and back upon the second application of the gate.

Quite often the physical capabilities do not allow us to create equal amplitude fields $\hat{Z}$ and $\hat{X}$ on demand. So we have to get by with limited control options. For example, consider a situation where field $\hat{Z}$ is always on (can't do anything about it) but we can turn on and off a small field $\hat{X}$, that is $\omega_X = g \ll \omega_Z$. In this case we can try the following protocol for performing rotations of the qubit state all along the Bloch sphere. Suppose we make $\omega_X = g$ for some time τ_1 and then we turn it off abruptly for some time τ_2, and repeat the sequence periodically.

Exercise 23: Consider the evolution of the qubit starting from state $|0\rangle$ during the time τ_1 when $\omega_X = g$. Show that the evolution is a rotation of the qubit state vector in the Bloch sphere, around the axis defined by components $(g,\, 0,\, \omega_Z)$, and with a constant angular velocity $\sqrt{\omega_Z^2 + g^2}$. What's the period of such rotation?

Exercise 24: Now suppose we set $\tau_1 = \pi/\sqrt{\omega_Z^2 + g^2}$ and $\tau_2 = \pi/\omega_Z$, and say $g = \omega_Z/10$. Show geometrically the evolution of the qubit states after each time τ_1 and τ_2 in the sequence. This you can actually do by hand but try QuTiP as well. Approximately what time would it take for the state vector to cross the equatorial plane?

Exercise 25: Time for a true QuTiP experiment. Let us set the sequence such that during the time τ we get $\omega_X = g$ and during the next interval τ we get $\omega_X = -g$. A "square wave" signal for ω_X. Let's further assume that $g = \omega_Z/1000$ and $\tau = \pi/\omega_Z$. Simulate the evolution of the qubit state. Plot $\langle Z \rangle$, $\langle X \rangle$, $\langle Y \rangle$ as a function of time.

Exercise 26: At what time in the previous exercise the evolution comes back to state $|1\rangle$ (for the first time). At what time the state becomes $|+\rangle$? (for the first time).

Exercise 27: If you succeeded with the previous exercises, try varying the time τ by around 0.1-1%. How does the evolution change? Look at you, you're becoming a real quantum engineer!

In real qubits it is usually hard to switch off the $\hat{Z}$-field, so no matter what else we do with it, the qubit state's evolution involves a rotation around the $\hat{Z}$ axis. We can "unwind this rotation" (often terminology is "unwind the Z-phase") by literally rotating ourselves around the Z-axis at the right angular velocity. Mathematically, the frame change transformation to unwind the Z-rotation is given by the unitary $\hat{U} = \exp(-i\omega_Z t\hat{Z}/2)$.

Exercise 28: Apply operator $\hat{U} = \exp(-i\omega_Z t\hat{Z}/2)$ to your solution of the previous exercise and plot the values of $\langle X \rangle$, $\langle Y \rangle$, $\langle Z \rangle$. Are the plots a bit simpler now?

F. Implementing quantum gates: evolution under periodic drive

A more practical method to rotate the qubit state in the Bloch sphere is to apply a periodic drive - a field along one axis oscillating with a special (resonance) frequency. Here comes the harmonic oscillator again (but not quite). We will come back to discussing specific physical implementation of qubits and their drives a bit later in the course. So for now let us consider a driven qubit with a simplest Hamiltonian

$$\hat{H}/\hbar = -\omega\hat{Z}/2 - g\cos(\omega_d t)\hat{X} \tag{9}$$

Let us remind that in the absence of drive ($g = 0$) the eigenstates are $|0\rangle$ (the eigenvalue E_0) and $|1\rangle$ (the eigenvalue E_1), such that $E_1 - E_0 = \hbar\omega$. So the two level system has a frequency associated with it. We already know the meaning of this frequency: any state that is a superposition of $|0\rangle$ and $|1\rangle$ rotates along the Z-axis at an angular frequency ω and hence the period $2\pi/\omega$. So perhaps something interesting happens if the drive frequency ω_d becomes close to ω. Starting to resemble some more harmonic oscillators?

Formally, the Hamiltonian function is now time-dependent. What do we do in this case? Well, we can always just duly solve the original Schrodinger equation (1) written in the computational basis $|0\rangle$ and $|1\rangle$. We take the most general qubit state $|\Psi\rangle = \alpha_0(t)|0\rangle + \alpha_1(t)|1\rangle$ with unknown and time-dependent amplitudes, and insert it into Eq. 1.

$$(\dot{\alpha_0}|0\rangle + \dot{\alpha_1}|1\rangle = i(\omega/2)\alpha_0|0\rangle - i(\omega/2)\alpha_1|1\rangle + ig\cos(\omega_d t)(\alpha_0|1\rangle + \alpha_1|0\rangle).$$

$$(\dot{\alpha_0} - i(\omega/2)\alpha_0 - ig\cos(\omega_d t)\alpha_1)|0\rangle + (\dot{\alpha_1} + i(\omega/2)\alpha_1 - ig\cos(\omega_d t)\alpha_0)|1\rangle = 0$$

Since $|0\rangle$ and $|1\rangle$ are the basis vectors, the above is only possible if the amplitudes in front of $|0\rangle$ and $|1\rangle$ are both 0. Equating those to zero gives us the following equations for the state of a periodically driven qubit:

$$\dot{\alpha_0} = +\alpha_0 \times (i\omega/2) + \alpha_1 \times ig\cos(\omega_d t)$$
$$\dot{\alpha_1} = -\alpha_1 \times (i\omega/2) + \alpha_0 \times ig\cos(\omega_d t)$$

So, how do we solve this? Let's quickly make sense of the case $g = 0$, no drive. Then $\alpha_0(t) = \alpha_0(t = 0)\exp(i\omega t/2)$, $\alpha_1(t) = \alpha_1(t = 0)\exp(-i\omega t/2)$ and assuming at $t = 0$ we had $\Psi(t = 0) = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$, we get $|\Psi(t) = \exp(i\omega t/2)(\cos(\theta/2)|0\rangle + \sin(\theta/2)\exp(-i\omega t)|1\rangle)$, indeed, a rotation of the Bloch vector around the Z-axis at frequency ω, as expected. The global phase factor $\exp(i\omega t/2)$ is irrelevant and can be dropped.

Exercise 29: Solve the above equations numerically and reconstruct the evolution of the Bloch state vector. Use initially $\omega_d = \omega$ and $g = \omega/1000$ and try to vary both. Start from various initial states, such as states $|0\rangle$, $|1\rangle$, $|i\rangle$, $|-i\rangle$, and see if they evolve into each other. Look, you are a quantum engineer again!

Exercise 30: Apply the drive frame operator $\hat{U} = \exp(-i\omega_d tZ/2)$ to the obtained numerical solution and see how it simplifies again. The meaning of this transformation is that you are now rotating along the Z-axis with an angular frequency ω_d

A more professional way to look at the evolution of a driven qubit is to go into the drive frame right away using the transformation $\hat{U} = \exp(-i\omega_d t\hat{Z}/2)$. In doing so, we obtain a new Hamiltonian,

$$\hat{H}^{\text{drive frame}} = -(\omega - \omega_d)\hat{Z}/2 - \exp(-i\omega_d t\hat{Z}/2)\hat{X}\exp(+i\omega_d t\hat{Z}/2) \times g\cos(\omega_d t)$$

Exercise 31: By applying that scary beast to $|0\rangle$ and $|1\rangle$, show that

$$\exp(-i\omega_d t\hat{Z}/2)\hat{X}\exp(+i\omega_d t\hat{Z}/2) = \begin{pmatrix} 0 & \exp(-i\omega_d t) \\ \exp(+i\omega_d t) & 0 \end{pmatrix}$$

Exercise 32: Now show that $\exp(-i\omega_d t\hat{Z}/2)\hat{X}\exp(+i\omega_d t\hat{Z}/2) \times g\cos(\omega_d t) =$

$$= (g/2)\hat{X} + (g/2)\begin{pmatrix} 0 & \exp(-2i\omega_d t) \\ \exp(+2i\omega_d t) & 0 \end{pmatrix}$$

So, the time dependent drive term in the original Hamiltonian got replaced by a familiar static term $g\hat{X}/2$ and an additional new drive term, which corresponds to driving the qubit at frequency $2\omega_d$. And remember, in the rotating frame, the qubit now has a frequency $\omega - \omega_d$, so there is a huge mismatch between the qubit transition frequency and the drive frequency. Perhaps this new drive term can be just ignored? Let's go with this guess for now. In this case we arrive at the following approximate Hamiltonian in the rotating frame:

$$\hat{H}^{\text{drive frame}} \approx -(\omega - \omega_d)\hat{Z}/2 - g\hat{X}/2.$$

This Hamiltonian is familiar from section E. In particular, let's consider the case $\omega - \omega_d = 0$, that is the resonant drive.

Exercise 33: At $\omega = \omega_d$ (resonant drive) the evolution of the qubit state in the drive frame is a rotation around $\hat{X}$. What is the period of such rotation?

Exercise 34: Recover the rotation of the Bloch sphere vector in the lab frame by applying the U^{-1} operator to the drive frame solution. Show the resulting trajectory on the Bloch sphere.

Exercise 35: Show the qubit state vector evolution along the Bloch sphere (in the drive frame), in case $\omega - \omega_d = g$, $\omega - \omega_g = 10g$, and $\omega - \omega_g = 100g$. Do you see why highly off-resonant drive of the qubit does not matter?

Exercise 36: Make a density plot of $\langle Z \rangle(t)$ as a function of $\omega - \omega_d$, for $g = 1$, $\omega - \omega_d$ between -3 and 3, and choose range for t such that several oscillations are seen. This is often called the "Chevron plot".

Exercise 37: Do the same plot as above but this time as a function of g for $\omega = \omega_d$. Also plot the frequency of oscillations as a function of g?
We have seen that applying an oscillating field along $\hat{X}$ at a frequency $\omega_d = \omega_Z = \omega$ implements a rotation of the Bloch vector around X-axis at the angular velocity g (in the drive frame). This rotation is called Rabi oscillations. One can repeat the reasoning in case the oscillating field is applied along the Y-axis, in which case the resulting qubit state rotation is also around the Y axis.

It is interesting to point out that a resonantly driven qubit behaves quite differently from a resonantly driven harmonic oscillator. The latter gains energy and increases the amplitude of oscillations indefinitely. The former only has two energy levels, and hence only one quantum of energy $\hbar\omega$ to gain. So instead of "swinging" stronger, the qubit state rotates faster (see previous exercise result). The frequency of this rotation is called Rabi frequency.

Congratulations. Having worked out these exercises you now know everything there is to know about a single qubit! You know how to deal with it, you know what can happen to it, and most importantly, you know how to create arbitrary rotations of the qubit state, which in quantum computing are called quantum gates.